Tilt-Prioritized Quadrocopter Attitude Control

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***Abstract*— This paper presents an attitude tracking control law and control allocation strategy for quadrocopters which prioritizes the vehicle's ability to achieve a desired translational acceleration. The quadrocopter's attitude error is split into a reduced attitude error, which describes the misalignment of the thrust direction, and a yaw error, which describes the orientation error about the thrust direction. A model-based proportional–derivative control law is derived, where the proportional action is in terms of the reduced attitude and yaw errors, and the derivative action is in terms of the quadrocopter's angular velocity error. Almost global asymptotic convergence of the reduced attitude error is established, and the convergence of the yaw error is proven. It is further shown that the attitude control law decouples the reduced attitude error dynamics from the yaw error dynamics. A control allocation strategy is derived which exploits the decoupling in order to prioritize the correction of the reduced attitude over the yaw error. The proposed control strategy is computationally lightweight and therefore well-suited to run on board quadrocopters at high rates. Experimental results demonstrate improved error recovery and position tracking performance.**

***Index Terms*— Aerial robotics, attitude control, control allocation, nonlinear control, quadrocopter.**

I. Introduction

QUADROCOPTERS have gained popularity over the past decade due to their exceptional mobility and mechanical simplicity, and numerous strategies for their control have been developed (see [1]–[6]). Driven by the increasing use of quadrocopters for various applications, such as aerial inspection, package delivery, or entertainment, quadrocopters are expected to encounter and recover from an increasingly large set of potential disturbances and perform increasingly aggressive flight maneuvers. However, today's state-of-the-art quadrocopter control strategies often struggle with recovering from large position and attitude errors or with tracking aggressive maneuvers, where the quadrocopter is close to its physical limits.

A. Motivation

Quadrocopters are inherently underactuated and can only generate a thrust in a single direction perpendicular to their rotor disks. As a result, their position and attitude dynamics are coupled and only their position and yaw orientation, i.e., the rotation about their thrust direction, can be independently controlled. Due to the coupled position and attitude dynamics, quadrocopter control strategies are often based on a cascaded control scheme, as shown in Fig. 1, consisting of an outer position control loop, an inner attitude control loop, and a control allocation algorithm: The position controller computes the thrust vector required to track the desired maneuver, the attitude controller computes the necessary torques to rotate the quadrocopter such that the desired thrust can be achieved, i.e., such that the rotor disks are aligned perpendicular to the desired thrust direction, and finally, the control allocation algorithm coordinates the four rotors such that they generate the desired thrust and body torques. The employed attitude control and control allocation algorithm therefore play important roles in a quadrocopter's control strategy. A desired maneuver can only be tracked accurately if the quadrocopter is oriented correctly and the rotors generate the desired thrust and body torques. When tracking aggressive maneuvers or recovering from large errors, the desired thrust and body torques can often not be met due to saturation limits of the individual rotor thrusts. If these saturation constraints are not properly considered in the control allocation, the performance of the overall control strategy may be significantly degraded or the system may even become unstable. A naive solution to ensure feasible rotor thrust commands is to clip the rotor thrusts corresponding to the desired collective thrust and body torques at their saturation limits. However, by clipping the rotor thrusts neither the desired collective thrust nor the desired body torques are produced exactly, and their different importance for the quadrocopter's stability and trajectory tracking performance is neglected. For example, a quadrocopter's position dynamics are invariant to its yaw orientation, rendering the yaw torque used to control its yaw orientation less important than its collective thrust.

This paper therefore addresses the attitude control and control allocation problem with the objective of prioritizing the alignment of the quadrocopter's thrust direction in order to

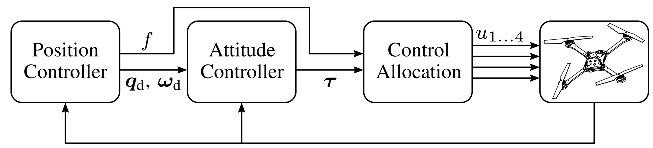

Fig. 1. Typical cascaded control scheme consisting of an outer position control loop and an inner attitude control loop. The position controller computes a desired thrust and attitude. The desired attitude is fed into an attitude controller that computes the required body torques, which are together with the desired collective thrust mapped to individual rotor thrusts.

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improve its trajectory tracking and error recovery performance. It is assumed that a position controller exists that provides a desired collective thrust and attitude.

B. Related Work

The attitude control problem for rigid bodies has been thoroughly investigated since the 1950s (see [7] and [8] and references therein) and can be divided into two categories: the full attitude control problem and the reduced attitude control problem. The objective of the former is to align a body-fixed coordinate frame with a reference frame whereas the objective of the latter is solely to point a body-fixed vector in a specified direction in a reference frame (and rotations about the specified direction are irrelevant). Since the position dynamics of a quadrocopter are fully determined by its thrust, which is constrained to a single direction with respect to the quadrocopter, an attitude controller that solves the reduced attitude control problem would be sufficient to control its position. Nonetheless, many applications also demand for controlling a quadrocopter's yaw orientation, for example, inspection, where data about the environment is typically gathered using a directional sensor. For this reason, various full attitude control techniques have been developed for the attitude stabilization and tracking control of quadrocopters, including a classical proportional–integral–derivative (PID) control approach assuming simplified dynamics [9], nonlinear proportional-derivative (PD) controllers based on rotation matrices [6], [10], unit quaternions [11], [12] or rotation vectors [13], [14], back-stepping and sliding-mode techniques [15]–[18] as well as optimal control methods such as linear quadratic regulators [9], [19] or model predictive control [20].

The problem of control allocation is often discussed in the context of overactuated systems where there are more actuators than strictly needed to meet the desired motion control objectives [21]. However, control allocation is also important for underactuated systems such as quadrocopters in the case of rotor failure [22], [23] or when the desired collective thrust and body torques require rotor thrusts that are beyond the rotors' physical limits. In the latter case, if no feasible rotor thrusts exist that generate the desired collective thrust and body torques, the control allocation algorithm has to degrade its performance and search for feasible rotor thrusts that minimize the control allocation error. In [24], two strategies to avoid rotor thrust saturations are analyzed; one based on a weighted least squares solution and another based on projecting the desired collective thrust and body torques onto their feasible set while maintaining their direction. In [25], a control allocation strategy is proposed that subsequently constrains the rotor thrusts that violate the saturation limits while prioritizing the roll and pitch torques over the commanded collective thrust and yaw torque.

The design of an attitude control law and control allocation algorithm is often done independent of each other, although the control allocation can significantly affect a quadrocopter's behavior and even destabilize it [21]. This can be overcome by a model predictive controller as proposed in [20] which handles the rotor thrust constraints directly in the attitude control algorithm. However, model predictive controllers are computationally expensive and can typically not be run at sufficiently high rates on the low-cost microcontrollers usually found on quadrocopters.

C. Contribution

This paper presents an attitude tracking control law and control allocation strategy for quadrocopters that seamlessly switch between full attitude and reduced attitude control, i.e., only controlling the direction of thrust. The main contribution of this paper is the design of an attitude tracking control law that decouples the alignment of the vehicle's thrust direction from the yaw orientation, and a control allocation strategy that exploits the decoupling in order to increase the control performance when the available control authority becomes scarce.

A quaternion-based PD control law is derived, where the proportional action is in terms of the reduced attitude and yaw errors, and the derivative action is in terms of the quadrocopter's angular velocity error. Almost global asymptotic stability of the reduced attitude error is formally established and convergence of the yaw error is shown. By splitting the attitude error into a reduced attitude error and a yaw error, the quadrocopter's thrust direction is steered along the shortest angular path toward the desired thrust direction, and as a consequence, the reduced attitude error typically converges faster and causes less position error than with a conventional controller based directly on the full attitude error. Furthermore, the proposed control law decouples the reduced attitude error dynamics from the yaw error dynamics; the reduced attitude error dynamics are independent of the yaw error, yaw angular velocity, and yaw torque.

A control allocation strategy is developed for the case when the desired collective thrust and torques cannot be met due to saturation limits of the rotors. The control allocation strategy prioritizes the roll and pitch torque that are required to rotate the quadrocopter's thrust direction into the desired direction over achieving the desired collective thrust and yaw torque. Because the derived attitude control law renders the reduced attitude dynamics independent of the applied yaw torque, the yaw torque can be constrained if the desired collective thrust and torques are not attainable without affecting the stability or convergence of the reduced attitude error dynamics and equivalently the position dynamics. Instead of wasting the available rotor thrust to generate a yaw torque that is not crucial for the quadrocopter's stability and position trajectory tracking performance, more control effort can be spent on achieving the desired roll and pitch torques and collective thrust.

The derived attitude tracking control law and control allocation strategy are both computationally light-weight and are thus well-suited to run at high rates on board a quadrocopter where computational resources are limited. The performance of the proposed control strategy is evaluated experimentally and a comparison with a conventional control strategy is provided. The proposed control strategy demonstrates enhanced position tracking performance for aggressive maneuvers and increased robustness when recovering from large errors, but at the expense of a slower yaw error response.

D. Outline

The remainder of this paper is structured as follows. Section II introduces the attitude representation used throughout this paper. A model of the quadrocopter dynamics is then presented in Section III. An attitude tracking control law is proposed in Section IV and its stability is discussed subsequently. Section V presents a strategy for allocating the rotor thrusts to meet the desired collective thrust and body torques. An experimental evaluation of the attitude tracking control law and control allocation strategy is given in Section VI, and conclusions are drawn in Section VII.

II. Attitude Representation

A survey of different attitude representations and their consequences for attitude control is presented in [8]. In this paper, the attitude is parametrized using unit quaternions as they use the least number of parameters (four) to represent the attitude globally in a singularity-free way. In the following, the most important properties of unit quaternions are introduced, and a more extensive list can be found in [26].

Let two coordinate systems $\mathcal{F}_1$ and $\mathcal{F}_2$ be separated by a rotation about a unit vector $\boldsymbol{n} \in \mathbb{S}^2$ by angle $\varphi \in \mathbb{R}$, where $\mathbb{S}^2$ denotes the two-sphere $\mathbb{S}^2 = \{\boldsymbol{n} \in \mathbb{R}^3 \mid \boldsymbol{n}^T\boldsymbol{n} = 1\}$. The attitude of $\mathcal{F}_2$ relative to $\mathcal{F}_1$ can then be described by a unit quaternion $\boldsymbol{q}$, consisting of a scalar q_0 and a vector $\tilde{\boldsymbol{q}} = (q_1, q_2, q_3)$, and is defined as

$$\boldsymbol{q} = \begin{bmatrix} q_0 \\ \tilde{\boldsymbol{q}} \end{bmatrix} = \begin{bmatrix} \cos\left(\frac{\varphi}{2}\right) \\ \boldsymbol{n}\sin\left(\frac{\varphi}{2}\right) \end{bmatrix}. \tag{1}$$

Consequently, the inverse quaternion representing the attitude of $\mathcal{F}_1$ relative to $\mathcal{F}_2$ is given by $\boldsymbol{q}^{-1} = (q_0, -\tilde{\boldsymbol{q}})$. In addition, it is apparent from (1) that $\boldsymbol{q} \in \mathbb{S}^3$, where $\mathbb{S}^3 = \{\boldsymbol{q} \in \mathbb{R}^4 \mid \boldsymbol{q}^T\boldsymbol{q} = 1\}$.

Let $\boldsymbol{q}_{\mathcal{F}_1}$ and $\boldsymbol{q}_{\mathcal{F}_2}$ describe the attitudes of $\mathcal{F}_1$ and $\mathcal{F}_2$ with respect to a common coordinate frame and let $\boldsymbol{q}$ represent the attitude of $\mathcal{F}_2$ with respect to $\mathcal{F}_1$, then it holds that

$$\boldsymbol{q}_{\mathcal{F}_2} = \boldsymbol{q} \otimes \boldsymbol{q}_{\mathcal{F}_1} \tag{2}$$

$$= \begin{bmatrix} q_0 & -\tilde{\boldsymbol{q}}^T \\ \tilde{\boldsymbol{q}} & q_0\boldsymbol{I} - [\tilde{\boldsymbol{q}}]_\times \end{bmatrix} \boldsymbol{q}_{\mathcal{F}_1} \tag{3}$$

where $\otimes$ denotes the quaternion multiplication operator, $\boldsymbol{I} \in \mathbb{R}^{3\times3}$ is the identity matrix, and $[\tilde{\boldsymbol{q}}]_\times$ is the skew-symmetric cross product matrix representation of $\tilde{\boldsymbol{q}}$

$$[\tilde{\boldsymbol{q}}]_\times = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix}. \tag{4}$$

The identity element of quaternion multiplication is given by $\boldsymbol{q}_I = (1, 0, 0, 0)$, i.e., a rotation with zero rotation angle, in which case $\boldsymbol{q} \otimes \boldsymbol{q}_I = \boldsymbol{q}_I \otimes \boldsymbol{q} = \boldsymbol{q}$.

The rotation matrix $\boldsymbol{R}(\boldsymbol{q}) \in \mathrm{SO}(3)$ corresponding to the rotation embodied by $\boldsymbol{q}$, i.e., the rotation from $\mathcal{F}_1$ to $\mathcal{F}_2$, is computed as

$$\boldsymbol{R}(\boldsymbol{q}) = (q_0^2 - \tilde{\boldsymbol{q}}^T\tilde{\boldsymbol{q}})\boldsymbol{I} + 2(\tilde{\boldsymbol{q}}\tilde{\boldsymbol{q}}^T - q_0[\tilde{\boldsymbol{q}}]_\times). \tag{5}$$

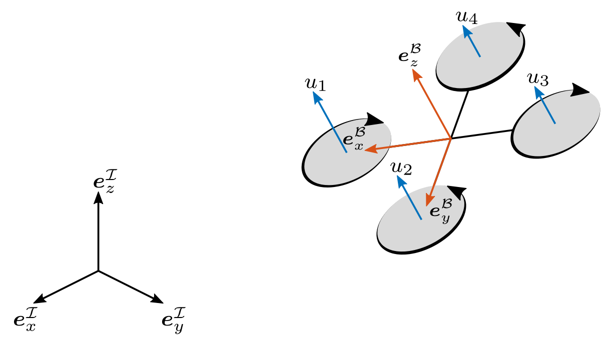

Fig. 2. Illustration of a quadrocopter with a body-fixed coordinate frame $\mathcal{B}$ and an inertial coordinate frame $\mathcal{I}$. The control inputs to the quadrocopter are the four rotor thrusts u_i that act along the quadrocopter's z-axis.

It is important to note that the space of unit quaternions $\mathbb{S}^3$ double covers the space of physical attitudes $\mathrm{SO}(3)$ and each pair of antipodal unit quaternions $\pm\boldsymbol{q} \in \mathbb{S}^3$ represents the same physical attitude. This implies that an attitude controller needs to stabilize a disconnected set of equilibrium points in $\mathbb{S}^3$ in order to avoid the unwinding phenomena [27], i.e., that the body to be controlled unnecessarily performs a rotation with a rotation angle larger than 180° to reach the desired attitude.

III. System Dynamics

This section presents the position and attitude dynamics of a quadrocopter with its rotors arranged as shown in Fig. 2. The quadrocopter is modelled as a rigid body with its position $\boldsymbol{p} = (p_x, p_y, p_z)$ measured in an inertial coordinate frame $\mathcal{I}$ and with its attitude parameterized by a unit quaternion $\boldsymbol{q}$ that represents the rotation from the inertial coordinate frame $\mathcal{I}$ to a body-fixed coordinate frame $\mathcal{B}$.

A. Control Inputs

The control inputs to the quadrocopter are the four rotor thrusts $\boldsymbol{u} = (u_1, u_2, u_3, u_4)$ as illustrated in Fig. 2.

The rotor thrusts are proportional to the rotors' angular velocities squared [28], which are tracked by high-bandwidth controllers on board the vehicle. Experimental results have shown very fast response times of the rotors to set point changes in the desired angular velocities (on the order of 15 ms). It is therefore assumed that the angular velocities of the rotors and equivalently the rotor thrusts can be set instantaneously, and that the rotor dynamics can be ignored.

The rotor thrusts are assumed to be subject to saturations. Each rotor thrust is limited by a minimum and maximum thrust

$$0 < u_{\min} \leq u_i \leq u_{\max} \tag{6}$$

where the lower limit is motivated by the minimum angular velocity of the rotor required for its controller to function properly [29], and the upper limit can be due to the maximum available voltage that can be applied to the rotor or the rotor's heat dissipation capacity. The set of feasible control inputs $\mathbb{U}$ can therefore be expressed as

$$\mathbb{U} = \{\boldsymbol{u} \in \mathbb{R}^4 \mid u_{\min}\boldsymbol{1} \preceq \boldsymbol{u} \preceq u_{\max}\boldsymbol{1}\} \tag{7}$$

where $\preceq$ denotes the componentwise inequality and $\boldsymbol{1}$ is a vector of ones.

The four rotor thrusts u_i collectively generate a thrust f along the vehicle's z-axis and body torques $\boldsymbol{\tau} = (\tau_x, \tau_y, \tau_z)$ about the vehicle's body-fixed coordinate axes. Typically, a quadrocopter's position and attitude controller are designed with the collective thrust f and torques $\boldsymbol{\tau}$ as inputs instead of the rotor thrusts $\boldsymbol{u}$. Hence, a virtual control input $\boldsymbol{v}$ is defined to be $\boldsymbol{v} := (f, \boldsymbol{\tau})$. The relationship between the actual control inputs $\boldsymbol{u}$ and the virtual control inputs $\boldsymbol{v}$ is described by

$$\boldsymbol{v} = \boldsymbol{B}\boldsymbol{u} \tag{8}$$

where

$$\boldsymbol{B} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & l & 0 & -l \\ -l & 0 & l & 0 \\ \kappa & -\kappa & \kappa & -\kappa \end{bmatrix} \tag{9}$$

l denotes the quadrocopter's arm length, and κ is the rotor specific thrust-to-drag ratio.

B. Equations of Motion

If aerodynamic effects such as drag on the quadrocopter's fuselage are neglected, then the position dynamics are determined by the orientation of the quadrocopter's z-axis and the total thrust generated by the four rotors

$$m\ddot{\boldsymbol{p}} = \boldsymbol{R}(\boldsymbol{q})^{-1} \begin{bmatrix} 0 \\ 0 \\ f \end{bmatrix} - m\boldsymbol{g} \tag{10}$$

where m denotes the quadrocopter's mass and $\boldsymbol{g}$ is the acceleration due to gravity. The attitude dynamics of the quadrocopter in quaternion space are given by [26]

$$\dot{\boldsymbol{q}} = \frac{1}{2} \begin{bmatrix} -\tilde{\boldsymbol{q}}^T \\ q_0 \boldsymbol{I} + [\tilde{\boldsymbol{q}}]_\times \end{bmatrix} \boldsymbol{\omega} \tag{11}$$

$$\boldsymbol{J}\dot{\boldsymbol{\omega}} = \boldsymbol{J}\boldsymbol{\omega} \times \boldsymbol{\omega} + \boldsymbol{\tau} \tag{12}$$

where $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$ denotes the quadrocopter's angular velocity and $\boldsymbol{J} \in \mathbb{R}^{3\times3}$ is the rotational inertia matrix, both expressed in the body-fixed frame $\mathcal{B}$. Without loss of generality, it is assumed that the principle axes of inertia coincide with the coordinate frame $\mathcal{B}$ such that $\boldsymbol{J} = \mathrm{diag}(J_{xx}, J_{yy}, J_{zz})$ and that the moments of inertia about the roll- and pitch-axes are equal, i.e., $J_{xx} = J_{yy}$.

IV. Attitude Control

In this section, a control law for tracking a desired attitude trajectory $\boldsymbol{q}_\mathrm{d}(t)$ with corresponding angular velocity $\boldsymbol{\omega}_\mathrm{d}(t)$ and angular acceleration $\dot{\boldsymbol{\omega}}_\mathrm{d}(t)$ is presented. First, the errors associated with tracking the desired attitude trajectory are defined, and then, a control law based on the body torques $\boldsymbol{\tau}$ is designed.

A. Attitude Tracking Errors

For a given desired attitude $\boldsymbol{q}_\mathrm{d}$ and current attitude $\boldsymbol{q}$, the attitude error is defined as the rotation from the current attitude to the desired one

$$\boldsymbol{q}_\mathrm{e} = \boldsymbol{q}_\mathrm{d} \otimes \boldsymbol{q}^{-1}. \tag{13}$$

In the remainder of this paper, it will be assumed that the components of the quaternion error are $\boldsymbol{q}_\mathrm{e} = (q_0, q_1, q_2, q_3)$. The angular velocity error is defined as the difference between the desired and current angular velocity

$$\boldsymbol{\omega}_\mathrm{e} = \bar{\boldsymbol{\omega}}_\mathrm{d} - \boldsymbol{\omega} \tag{14}$$

where $\boldsymbol{\omega}_\mathrm{e}$ is expressed in the body-fixed coordinate frame $\mathcal{B}$, and consequently, $\bar{\boldsymbol{\omega}}_\mathrm{d}$ is the desired angular velocity expressed in the body-fixed frame $\mathcal{B}$, i.e., $\bar{\boldsymbol{\omega}}_\mathrm{d} = \boldsymbol{R}(\boldsymbol{q}_\mathrm{e})^{-1}\boldsymbol{\omega}_\mathrm{d}$. The dynamics of the attitude and angular velocity errors can be computed by taking their time derivative and using (11) and (12)

$$\dot{\boldsymbol{q}}_\mathrm{e} = \frac{1}{2} \begin{bmatrix} -\tilde{\boldsymbol{q}}_\mathrm{e}^T \\ q_0 \boldsymbol{I} - [\tilde{\boldsymbol{q}}_\mathrm{e}]_\times \end{bmatrix} \boldsymbol{\omega}_\mathrm{e} \tag{15}$$

$$\boldsymbol{J}\dot{\boldsymbol{\omega}}_\mathrm{e} = \boldsymbol{J}\dot{\bar{\boldsymbol{\omega}}}_\mathrm{d} - (\boldsymbol{J}\boldsymbol{\omega} \times \boldsymbol{\omega} + \boldsymbol{\tau}) \tag{16}$$

with $\dot{\bar{\boldsymbol{\omega}}}_\mathrm{d} = \boldsymbol{\omega}_\mathrm{e} \times \bar{\boldsymbol{\omega}}_\mathrm{d} + \boldsymbol{R}(\boldsymbol{q}_\mathrm{e})^{-1}\dot{\boldsymbol{\omega}}_\mathrm{d}$.

As the objective is to design a control law capable of prioritizing the alignment of the vehicle's thrust direction over correcting its yaw orientation, two further attitude errors are defined: a reduced attitude error $\boldsymbol{q}_{\mathrm{e,red}}$ and a yaw error $\boldsymbol{q}_{\mathrm{e,yaw}}$. The reduced attitude error is a measure of the misalignment of the quadrocopter's thrust direction and is defined as the shortest rotation that aligns the quadrocopter's current direction of thrust with the desired one, i.e.,

$$\boldsymbol{R}(\boldsymbol{q}_\mathrm{e})\boldsymbol{e}_z^{\mathcal{B}} = \boldsymbol{R}(\boldsymbol{q}_{\mathrm{e,red}})\boldsymbol{e}_z^{\mathcal{B}} \tag{17}$$

where $\boldsymbol{e}_z^{\mathcal{B}}$ is the quadrocopter's z-axis expressed in its body frame $\mathcal{B}$, i.e., $\boldsymbol{e}_z^{\mathcal{B}} = (0, 0, 1)$. Evaluating (17) and using that the last component of $\boldsymbol{q}_{\mathrm{e,red}}$ needs to be zero in order for $\boldsymbol{q}_{\mathrm{e,red}}$ to be the shortest rotation, the reduced attitude error can be computed to be

$$\boldsymbol{q}_{\mathrm{e,red}} = \frac{1}{\sqrt{q_0^2 + q_3^2}} \begin{bmatrix} q_0^2 + q_3^2 \\ q_0\ q_1 - q_2\ q_3 \\ q_0\ q_2 + q_1\ q_3 \\ 0 \end{bmatrix}. \tag{18}$$

It directly follows from (18) that the reduced attitude error is zero, i.e., $\boldsymbol{q}_{\mathrm{e,red}} = \boldsymbol{q}_I$, if $q_0^2 + q_3^2 = 1$. Note that if the desired z-axis points in the opposite direction of the quadrocopter's current z-axis, i.e., if $q_0^2 + q_3^2 = 0$, then $\boldsymbol{q}_{\mathrm{e,red}}$ is not well defined since any rotation with a rotation angle of 180° about any axis in the body-fixed xy-plane would align the z-axis correctly with minimal rotation angle.

The yaw error is defined as the subsequent rotation required such that also the direction of the vehicle's x- and y-axes are aligned with the desired coordinate frame, i.e., such that

$$\boldsymbol{q}_\mathrm{e} = \boldsymbol{q}_{\mathrm{e,yaw}} \otimes \boldsymbol{q}_{\mathrm{e,red}}. \tag{19}$$

Solving (19) for the yaw error yields

$$\boldsymbol{q}_{\mathrm{e,yaw}} = \frac{1}{\sqrt{q_0^2 + q_3^2}} \begin{bmatrix} q_0 \\ 0 \\ 0 \\ q_3 \end{bmatrix}. \tag{20}$$

A visualization of the reduced attitude error and yaw error is shown in Fig. 3.

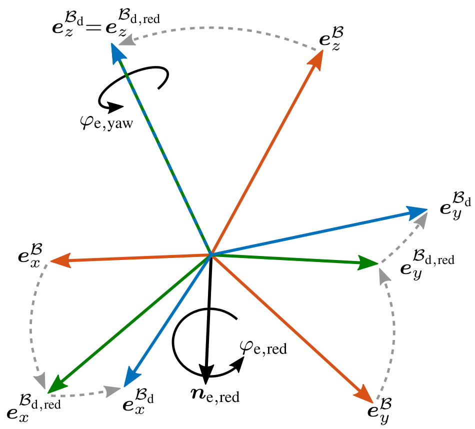

Fig. 3. Visualization of the rotations represented by the reduced attitude error $\boldsymbol{q}_\mathrm{e,red}$ and the yaw error $\boldsymbol{q}_\mathrm{e,yaw}$. The coordinate frame $\mathcal{B}$ represents the vehicle's current attitude $\boldsymbol{q}$ whereas the coordinate frame $\mathcal{B}_\mathrm{d}$ represents the desired attitude $\boldsymbol{q}_\mathrm{d}$. Rotating the $\mathcal{B}$-frame by $\boldsymbol{q}_\mathrm{e,red}$ (a rotation of $\varphi_\mathrm{e,red}$ about $\boldsymbol{n}_\mathrm{e,red}$) yields the intermediate coordinate frame $\mathcal{B}_\mathrm{d,red}$ whose z-axis coincides with the desired z-axis. A subsequent rotation by $\boldsymbol{q}_\mathrm{e,yaw}$ (a rotation of $\varphi_\mathrm{e,yaw}$ about the desired z-axis) then also aligns the x- and y-axis with the desired coordinate frame $\mathcal{B}_\mathrm{d}$.

B. Control Law

Consider the attitude error dynamics (15) and (16), and the control law

$$\boldsymbol{\tau} = k_{\mathrm{p},xy}\tilde{\boldsymbol{q}}_\mathrm{e,red} + k_{\mathrm{p},z}\mathrm{sgn}(q_0)\tilde{\boldsymbol{q}}_\mathrm{e,yaw} + \boldsymbol{K}_\mathrm{d}\boldsymbol{\omega}_\mathrm{e} + \boldsymbol{\tau}_\mathrm{ff} \quad (21)$$

where $\mathrm{sgn}\,(q_0)$ is defined as

$$\mathrm{sgn}(q_0) = \begin{cases} 1, & \text{if } q_0 \geq 0 \\ -1, & \text{if } q_0 < 0 \end{cases} \quad (22)$$

$\boldsymbol{K}_\mathrm{d} = \mathrm{diag}(k_{\mathrm{d},xy}, k_{\mathrm{d},xy}, k_{\mathrm{d},z})$, $\boldsymbol{\tau}_\mathrm{ff}$ consists of a feed-forward and a feedback linearization term

$$\boldsymbol{\tau}_\mathrm{ff} = \boldsymbol{J}\dot{\boldsymbol{\omega}}_\mathrm{d} - \boldsymbol{J}\boldsymbol{\omega} \times \boldsymbol{\omega} \quad (23)$$

and $k_{\mathrm{p},xy}, k_{\mathrm{p},z}, k_{\mathrm{d},xy}$, and $k_{\mathrm{d},z}$ are positive scalar constants. Then, the following properties hold:

1) The only equilibrium points of (15) and (16) are $(\boldsymbol{q}_\mathrm{e}, \boldsymbol{\omega}_\mathrm{e}) = (\pm\boldsymbol{q}_I, 0)$.
2) The reduced attitude error almost globally asymptotically converges to zero, i.e., $\boldsymbol{q}_\mathrm{e,red}(t) \rightarrow \boldsymbol{q}_I$ and $\boldsymbol{\omega}_{\mathrm{e},xy}(t) \rightarrow 0$ as $t \rightarrow \infty$ for almost all $(\boldsymbol{q}_\mathrm{e}(t_0), \boldsymbol{\omega}_\mathrm{e}(t_0)) \in \mathbb{S}^3 \times \mathbb{R}^3$, where $\boldsymbol{\omega}_{\mathrm{e},xy}$ is the vector consisting of the x- and y-components of $\boldsymbol{\omega}_\mathrm{e}$.
3) The yaw error asymptotically converges to zero in a neighborhood of $(\boldsymbol{q}_\mathrm{e,red}, \boldsymbol{\omega}_{\mathrm{e},xy}) = (\boldsymbol{q}_I, 0)$, i.e., $\boldsymbol{q}_\mathrm{e,yaw}(t) \rightarrow \pm\boldsymbol{q}_I$ and $\omega_{\mathrm{e},z}(t) \rightarrow 0$ as $t \rightarrow \infty$ for all $(\boldsymbol{q}_\mathrm{e}(t_0), \boldsymbol{\omega}_\mathrm{e}(t_0)) \in \mathbb{S}^3 \times \mathbb{R}^3$ with $(\boldsymbol{q}_\mathrm{e,red}(t_0), \boldsymbol{\omega}_{\mathrm{e},xy}(t_0))$ sufficiently close to $(\boldsymbol{q}_I, 0)$.
4) The yaw error almost globally converges to zero, i.e., $\boldsymbol{q}_\mathrm{e,yaw}(t) \rightarrow \pm\boldsymbol{q}_I$ and $\omega_{\mathrm{e},z}(t) \rightarrow 0$ as $t \rightarrow \infty$ for almost all $(\boldsymbol{q}_\mathrm{e}(t_0), \boldsymbol{\omega}_\mathrm{e}(t_0)) \in \mathbb{S}^3 \times \mathbb{R}^3$ but not necessarily in a Lyapunov sense.
5) Both equilibrium points $(\pm\boldsymbol{q}_I, 0)$ are stable.

Proof: It is straightforward to verify 1) by inserting the control law (21) into the error dynamics (15) and (16).

In order to prove 2), consider the Lyapunov candidate function

$$\mathcal{V}_1 = \frac{1}{2}\boldsymbol{\omega}_\mathrm{e}^T\boldsymbol{D}\boldsymbol{J}\boldsymbol{\omega}_\mathrm{e} + k_{\mathrm{p},xy}\left(1 - \sqrt{q_0^2 + q_3^2}\right) \quad (24)$$

with $\boldsymbol{D} = \mathrm{diag}(1, 1, 0)$. $\mathcal{V}_1$ is positive everywhere except at $(\boldsymbol{q}_\mathrm{e,red}, \boldsymbol{\omega}_{\mathrm{e},xy}) = (\boldsymbol{q}_I, 0)$, i.e., $q_0^2 + q_3^2 = 1$. The time derivative of $\mathcal{V}_1$ along the trajectories of the system is

$$\begin{aligned} \dot{\mathcal{V}}_1 &= -\boldsymbol{\omega}_\mathrm{e}^T\boldsymbol{D}(k_{\mathrm{p},xy}\tilde{\boldsymbol{q}}_\mathrm{e,red} + k_{\mathrm{p},z}\tilde{\boldsymbol{q}}_\mathrm{e,yaw} + \boldsymbol{K}_\mathrm{d}\boldsymbol{\omega}_\mathrm{e}) \\ &\quad + k_{\mathrm{p},xy}\tilde{\boldsymbol{q}}_\mathrm{e,red}^T\boldsymbol{\omega}_\mathrm{e} && (25) \\ &= -k_{\mathrm{d},xy}\boldsymbol{\omega}_{\mathrm{e},xy}^T\boldsymbol{\omega}_{\mathrm{e},xy} && (26) \\ &\leq 0. && (27) \end{aligned}$$

Due to the non-autonomous nature of the error dynamics (the desired attitude trajectory explicitly depends on time), LaSalle's invariance principle cannot be used to conclude convergence of $\mathcal{V}_1$ [30]. However, since $\dot{\mathcal{V}}_1$ is negative semi-definite, $\mathcal{V}_1(t) \leq \mathcal{V}_1(0)$ for $t \geq 0$, and hence, $\boldsymbol{\omega}_{\mathrm{e},xy}$ is bounded. The second time derivative of $\mathcal{V}_1$ is given by

$$\ddot{\mathcal{V}}_1 = -2\boldsymbol{\omega}_\mathrm{e}^T\boldsymbol{D}\boldsymbol{K}_\mathrm{d}\boldsymbol{J}^{-1}(k_{\mathrm{p},xy}\tilde{\boldsymbol{q}}_\mathrm{e,red} + k_{\mathrm{p},z}\tilde{\boldsymbol{q}}_\mathrm{e,yaw} + \boldsymbol{K}_\mathrm{d}\boldsymbol{\omega}_\mathrm{e}) \quad (28)$$

which is bounded because $\boldsymbol{\omega}_{\mathrm{e},xy}$ is bounded and $\boldsymbol{q}_\mathrm{e,red}$ is structurally bounded. Therefore, $\dot{\mathcal{V}}_1$ is uniformly continuous and by Barbalat's lemma, $\dot{\mathcal{V}}_1 \rightarrow 0$ as $t \rightarrow \infty$, which implies that the angular velocity error $\boldsymbol{\omega}_{\mathrm{e},xy}$ asymptotically approaches zero and $\mathcal{V}_1$ converges. By inserting the control law (21) into the angular velocity error dynamics (16), it becomes obvious that $\mathcal{V}_1$ must converge to zero since $\dot{\boldsymbol{\omega}}_{\mathrm{e},xy}$ only vanishes for $\boldsymbol{q}_\mathrm{e,red} = \boldsymbol{q}_I$. Note that although unit quaternions are a singularity-free attitude representation, a singularity at $q_0^2 + q_3^2 = 0$ has been introduced by splitting the attitude error $\boldsymbol{q}_\mathrm{e}$ into a reduced attitude error $\boldsymbol{q}_\mathrm{e,red}$ and a yaw error $\boldsymbol{q}_\mathrm{e,yaw}$. As a consequence of the singularity, the control law (21) and the time derivative of $\mathcal{V}_1$ are not defined when $q_0^2 + q_3^2 = 0$, and hence, the reduced attitude error only converges to zero for almost all $(\boldsymbol{q}_\mathrm{e}(t_0), \boldsymbol{\omega}_\mathrm{e}(t_0)) \in \mathbb{S}^3 \times \mathbb{R}^3$. This completes the proof of 2).

In order to prove 3), consider the Lyapunov-like candidate function

$$\mathcal{V}_2 = \frac{1}{2}\boldsymbol{\omega}_\mathrm{e}^T\boldsymbol{J}\boldsymbol{\omega}_\mathrm{e} + \begin{cases} 2k_{\mathrm{p},z}(1 - q_0), & \text{if } q_0 \geq 0 \\ 2k_{\mathrm{p},z}(1 + q_0), & \text{if } q_0 < 0 \end{cases} \quad (29)$$

which is positive everywhere except at $(\boldsymbol{q}_\mathrm{e}, \boldsymbol{\omega}_\mathrm{e}) = (\pm\boldsymbol{q}_I, 0)$. Furthermore, let $\mathbb{M}$ be a set defined as

$$\mathbb{M} := \left\{(\boldsymbol{q}_\mathrm{e}, \boldsymbol{\omega}_\mathrm{e}) \in \mathbb{S}^3 \times \mathbb{R}^3 | q_0^2 + q_3^2 = 1, \boldsymbol{\omega}_{\mathrm{e},xy} = 0\right\}. \quad (30)$$

Note that $\mathbb{M}$ is an invariant set with respect to attitude error dynamics (15) and (16), i.e., if $(\boldsymbol{q}_\mathrm{e}(t_0), \boldsymbol{\omega}_\mathrm{e}(t_0)) \in \mathbb{M}$, then $(\boldsymbol{q}_\mathrm{e}(t), \boldsymbol{\omega}_\mathrm{e}(t)) \in \mathbb{M}$ for all $t \geq t_0$. Suppose that $(\boldsymbol{q}_\mathrm{e}(t_0), \boldsymbol{\omega}_\mathrm{e}(t_0)) \in \mathbb{M}$, then, for any $t \geq t_0$ and $q_0 \neq 0$, the time derivative of $\mathcal{V}_2$ along the trajectories of the system is given by

$$\begin{aligned}\dot{\mathcal{V}}_2 &= -\boldsymbol{\omega}_{\mathrm{e}}^T(k_{\mathrm{p},xy}\tilde{\boldsymbol{q}}_{\mathrm{e,red}} + k_{\mathrm{p},z}\mathrm{sgn}(q_0)\tilde{\boldsymbol{q}}_{\mathrm{e,yaw}} + \boldsymbol{K}_{\mathrm{d}}\boldsymbol{\omega}_{\mathrm{e}}) \\ &\quad + \begin{cases} k_{\mathrm{p},z}\tilde{\boldsymbol{q}}_{\mathrm{e}}^T\boldsymbol{\omega}_{\mathrm{e}}, & \text{if } q_0 > 0 \\ -k_{\mathrm{p},z}\tilde{\boldsymbol{q}}_{\mathrm{e}}^T\boldsymbol{\omega}_{\mathrm{e}}, & \text{if } q_0 < 0 \end{cases} \qquad (31) \\ &= -\omega_{\mathrm{e},z}(k_{\mathrm{p},z}\mathrm{sgn}(q_0)q_3 + k_{\mathrm{d},z}\omega_{\mathrm{e},z}) \\ &\quad + \begin{cases} k_{\mathrm{p},z}q_3\omega_{\mathrm{e},z}, & \text{if } q_0 > 0 \\ -k_{\mathrm{p},z}q_3\omega_{\mathrm{e},z}, & \text{if } q_0 < 0 \end{cases} \qquad (32) \\ &= -k_{\mathrm{d},z}\omega_{\mathrm{e},z}^2 \qquad (33) \\ &\leq 0. \qquad (34)\end{aligned}$$

The time derivative of $\mathcal{V}_2$ is not defined for $q_0 = 0$, but since $\mathcal{V}_2$ is continuous when q_0 switches its sign, $\mathcal{V}_2$ is nonincreasing.[1] Following the same reasoning as for $\dot{\mathcal{V}}_1$ and invoking Barbalat's lemma, the angular velocity error $\omega_{\mathrm{e},z}$ asymptotically approaches zero and $\mathcal{V}_2$ converges. Again, by inserting (21) into (16), it follows that $\mathcal{V}_2$ must converge to zero and as a result, $\boldsymbol{q}_{\mathrm{e}} \to \pm\boldsymbol{q}_I$ and equivalently $\boldsymbol{q}_{\mathrm{e,yaw}} \to \pm\boldsymbol{q}_I$ as $t \to \infty$. Linearizing the error dynamics (15) and (16) about the set $\mathbb{M}$ reveals that the error dynamics of q_0, q_3, and $\omega_{\mathrm{e},z}$ are not affected by small perturbations of $(\boldsymbol{q}_{\mathrm{e,red}}, \boldsymbol{\omega}_{\mathrm{e},xy})$ about $(\boldsymbol{q}_I, 0)$. Therefore, $(\boldsymbol{q}_{\mathrm{e,yaw}}, \omega_{\mathrm{e},z})$ is asymptotically stable in a small neighborhood of $\mathbb{M}$.

It follows from 2) that $(\boldsymbol{q}_{\mathrm{e,red}}, \boldsymbol{\omega}_{\mathrm{e},xy})$ reaches the neighborhood of $\mathbb{M}$ in finite time, and consequently, $(\boldsymbol{q}_{\mathrm{e,yaw}}, \omega_{\mathrm{e},z})$ converges for almost all $(\boldsymbol{q}_{\mathrm{e}}(t_0), \boldsymbol{\omega}_{\mathrm{e}}(t_0)) \in \mathbb{S}^3 \times \mathbb{R}^3$, which establishes 4). Although $(\boldsymbol{q}_{\mathrm{e,yaw}}, \omega_{\mathrm{e},z})$ is locally asymptotically stable and almost globally converges to $(\pm\boldsymbol{q}_I, 0)$, the errors $(\boldsymbol{q}_{\mathrm{e,yaw}}, \omega_{\mathrm{e},z})$ are not almost globally stable in the sense of Lyapunov because they are not bounded during the finite time required to reach the neighborhood of $\mathbb{M}$.

Finally, 5) directly results from 1)–4). Since both $(\boldsymbol{q}_{\mathrm{e,red}}, \boldsymbol{\omega}_{\mathrm{e},xy})$ and $(\boldsymbol{q}_{\mathrm{e,yaw}}, \omega_{\mathrm{e},z})$ are locally asymptotically stable about $(\boldsymbol{q}_I, 0)$ and $(\pm\boldsymbol{q}_I, 0)$, respectively, it follows from (19) that $(\boldsymbol{q}_{\mathrm{e}}, \boldsymbol{\omega}_{\mathrm{e}}) = (\pm\boldsymbol{q}_I, 0)$ are stable equilibrium points, and hence, the control law (21) avoids the unwinding phenomena.

C. Discussion

1) Yaw Stability: The fact that the yaw error is only locally asymptotically stable does not affect the overall stability of the quadrocopter because the position dynamics only depend on the reduced attitude error.

2) Control Law Interpretation: The torque due to the reduced attitude error pushes the quadrocopter's z-axis along the shortest angular path to the desired z-axis, while the torque due to the yaw error induces a rotation about the quadrocopter's current z-axis. In contrast to conventional controllers whose proportional action is directly in terms of the full attitude error $\tilde{\boldsymbol{q}}_{\mathrm{e}}$ (see [11], [12], or similarly for rotation vectors [13], [14] or rotation matrices [6], [10]), which represents the shortest rotation between the desired and current attitude including the yaw error, the proposed controller steers the quadrocopter's z-axis along a shorter angular path to the desired direction, and the reduced attitude error therefore converges faster. A comparison for stabilizing an attitude error with the proposed control law (21) and with a conventional controller based on the full attitude error [33]

$$\boldsymbol{\tau} = \mathrm{sgn}(q_0)\boldsymbol{K}_{\mathrm{p}}\tilde{\boldsymbol{q}}_{\mathrm{e}} + \boldsymbol{K}_{\mathrm{d}}\boldsymbol{\omega}_{\mathrm{e}} + \boldsymbol{\tau}_{\mathrm{ff}} \qquad (35)$$

is shown in Fig. 4. The control gains $\boldsymbol{K}_{\mathrm{p}} = \mathrm{diag}(k_{\mathrm{p},xy}, k_{\mathrm{p},xy}, k_{\mathrm{p},z})$ and $\boldsymbol{K}_{\mathrm{d}}$ are chosen to be equal for both controllers such that the system response for small errors is identical. A list of all relevant parameters used to generate Fig. 4 can be found in Table I.

3) Decoupled Error Dynamics: The Lyapunov candidate function $\mathcal{V}_1$ is invariant to any yaw rotation and angular velocity about the quadrocopter's z-axis and therefore only captures the reduced attitude error dynamics. Its derivative $\dot{\mathcal{V}}_1$ is also invariant to any yaw rotation and angular velocity about the z-axis, and hence, it can be concluded that the attitude control law (21) decouples the reduced attitude error dynamics from the yaw error dynamics. In addition, $\dot{\mathcal{V}}_1$ does not depend on the yaw torque, and consequently, the reduced attitude can be fully controlled by the roll and pitch torque. Fig. 5 shows the results for stabilizing the same error as in Fig. 4, but with the yaw torque τ_z set to zero. Unlike controllers that depend on the full attitude error, the convergence of the reduced attitude is unaffected by the yaw torque constraint.

4) Controller Gains: Finding good controller gains can be a tedious task. One approach to simplify the tuning of the control gains is to analyze the system's behavior for small errors. If the error dynamics are linearized around their equilibria $(\pm\boldsymbol{q}_I, 0)$,

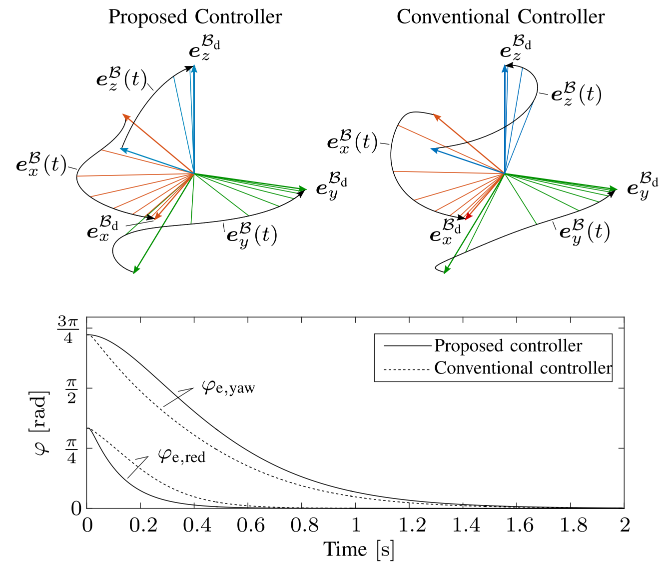

Fig. 4. Top plots show the results of reorienting the quadrocopter from rest with an initial attitude error of $\boldsymbol{q}_{\mathrm{e}} = (0.37, 0.31, -0.39, 0.78)$ to rest with the proposed attitude control law (21) and with a conventional attitude control law (35). The proposed control law realigns the quadrocopter's z-axis following the shortest rotation while simultaneously rotating about it in order to correct for errors in the yaw orientation. The plot at the bottom shows the angle errors φ of the reduced attitude error $\boldsymbol{q}_{\mathrm{e,red}}$ and yaw error $\boldsymbol{q}_{\mathrm{e,yaw}}$. The proposed controller corrects the reduced attitude error faster, but at the expense of a slower response in the yaw error.

[1]This can be rigorously shown by converting the error dynamics (15) and (16) to a hybrid system with a discrete state denoting the sign of q_0 (see [31]) and applying Lyapunov stability for hybrid systems [32], but is left out herein for the sake of brevity.

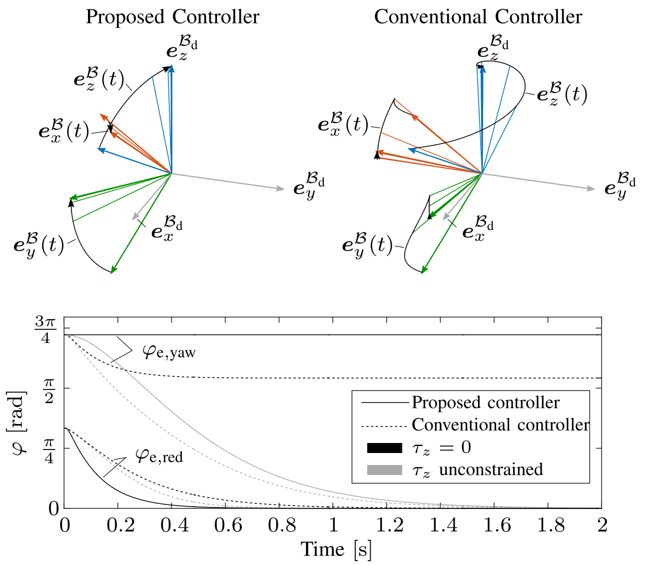

Fig. 5. Plots on the top show the results of reorienting the quadrocopter from rest for the same initial attitude error as in Fig. 4 but with the applied yaw torque τ_z set to zero. The bottom plot depicts the corresponding angle errors φ for the reduced attitude error and yaw error. The results in Fig. 4 with the yaw torque unconstrained are added for reference and colored gray. The response of the vehicle's z-axis remains unaffected with the proposed control law due to the decoupling of the reduced attitude error from the yaw error. For controllers based on the full attitude error, the yaw error is still partially corrected using roll and pitch torques, but the rate of convergence of the reduced attitude error changes and becomes slower.

then they can be written as

$$\ddot{\tilde{\boldsymbol{q}}}_{\mathrm{e}} = -\frac{1}{2}\boldsymbol{J}^{-1}\boldsymbol{K}_{\mathrm{p}}\tilde{\boldsymbol{q}}_{\mathrm{e}} - \boldsymbol{J}^{-1}\boldsymbol{K}_{\mathrm{d}}\dot{\tilde{\boldsymbol{q}}}_{\mathrm{e}}. \tag{36}$$

Therefore, by choosing the proportional gains according to

$$k_{\mathrm{p},xy} = \frac{2J_{xx}}{\sigma_{xy}^2} \quad \text{and} \quad k_{\mathrm{p},z} = \frac{2J_{zz}}{\sigma_z^2} \tag{37}$$

and the derivative gains according to

$$k_{\mathrm{d},xy} = \frac{2J_{xx}\zeta_{xy}}{\sigma_{xy}} \quad \text{and} \quad k_{\mathrm{d},z} = \frac{2J_{zz}\zeta_z}{\sigma_z} \tag{38}$$

the linearized system behaves like a second-order system with time constant σ_{xy} and damping ratio ζ_{xy} along the roll- and pitch-axes and with time constant σ_z and damping ratio ζ_z along the yaw axis, respectively.

V. Control Allocation

In this section, a method for allocating the available rotor thrusts to the desired collective thrust and torques is presented. Since the feasible control inputs $\boldsymbol{u}$ are constrained to the set $\mathbb{U}$, also the set of the virtual control inputs that are attainable from $\mathbb{U}$ is constrained. The set of attainable virtual control inputs $\mathbb{V}$ can be determined from the set of feasible control inputs (7) and the rotor thrust mapping (8)

$$\mathbb{V} = \{\boldsymbol{v} \in \mathbb{R}^4 \mid u_{\min}\mathbf{1} \preceq \boldsymbol{B}^{-1}\boldsymbol{v} \preceq u_{\max}\mathbf{1}\}. \tag{39}$$

If the desired virtual control inputs $\boldsymbol{v}$ from the position and attitude controller lie in the attainable set, i.e., $\boldsymbol{v} \in \mathbb{V}$, then a feasible control input $\boldsymbol{u} \in \mathbb{U}$ can simply be found by inverting (8)

$$\boldsymbol{u} = \boldsymbol{B}^{-1}\boldsymbol{v}. \tag{40}$$

However, if $\boldsymbol{v} \notin \mathbb{V}$, then no feasible control input $\boldsymbol{u} \in \mathbb{U}$ can be found such that the rotors collectively generate the desired virtual control input $\boldsymbol{v}$. In this case, in order to guarantee that the commanded rotor thrusts u_i are feasible, it is proposed to project the desired virtual control input $\boldsymbol{v}$ onto the boundary of $\mathbb{V}$ by prioritizing the virtual control inputs according to their importance for the stability and trajectory tracking performance of the quadrocopter.

A. Control Prioritization

Because the correct orientation of the quadrocopter's thrust direction is crucial for tracking a desired position trajectory and because in the design of the cascaded control scheme (see Fig. 1) it is typically assumed that the orientation of the quadrocopter's thrust direction can be changed quickly, the highest priority is given to achieving the desired roll and pitch torque τ_x and τ_y. Since both the position and the reduced attitude error dynamics are independent of the vehicle's yaw orientation, the second highest priority is given to achieving the desired collective thrust f and the least priority is given to the yaw torque τ_z. Consequently, the projection of $\boldsymbol{v}$ onto $\mathbb{V}$ is performed as follows.

First, it is ensured that the desired roll and pitch torque are attainable. Using the Fourier–Motzkin elimination [34] to project $\mathbb{V}$ onto the τ_x, τ_y-plane, the inequalities describing the set of attainable roll and pitch torques are found to be

$$\begin{bmatrix} -(u_{\max} - u_{\min})l \\ -(u_{\max} - u_{\min})l \end{bmatrix} \preceq \begin{bmatrix} \tau_x \\ \tau_y \end{bmatrix} \preceq \begin{bmatrix} (u_{\max} - u_{\min})l \\ (u_{\max} - u_{\min})l \end{bmatrix}. \tag{41}$$

The desired roll and pitch torque are, thus, constrained to

$$\hat{\tau}_x \leftarrow \operatorname{constrain}(\tau_x, -\tau_{\max,xy}, \tau_{\max,xy}) \tag{42}$$

$$\hat{\tau}_y \leftarrow \operatorname{constrain}(\tau_y, -\tau_{\max,xy}, \tau_{\max,xy}) \tag{43}$$

where $\tau_{\max,xy} = (u_{\max} - u_{\min})l$. Next, the set of attainable collective thrusts for the given torques $\hat{\tau}_x$ and $\hat{\tau}_y$ is determined. By inserting $\hat{\tau}_x$ and $\hat{\tau}_y$ into (39) and using the Fourier–Motzkin algorithm to eliminate τ_z, the set of attainable collective thrusts is found to be

$$\underbrace{4u_{\min} + \frac{|\hat{\tau}_x|}{l} + \frac{|\hat{\tau}_y|}{l}}_{=:f_{\min}} \leq f \leq \underbrace{4u_{\max} - \frac{|\hat{\tau}_x|}{l} - \frac{|\hat{\tau}_y|}{l}}_{=:f_{\max}}. \tag{44}$$

Hence, if the desired collective thrust f exceeds the lower or upper limit, it is clipped resulting in

$$\hat{f} \leftarrow \operatorname{constrain}(f, f_{\min}, f_{\max}). \tag{45}$$

Finally, the attainable set of yaw torques is found by inserting the adjusted torques $\hat{\tau}_x$ and $\hat{\tau}_y$ and the collective thrust $\hat{f}$ into (39)

$$\underbrace{\kappa\begin{bmatrix} 4u_{\min} - \hat{f} + 2\frac{|\hat{\tau}_y|}{l} \\ -4u_{\max} + \hat{f} + 2\frac{|\hat{\tau}_x|}{l} \end{bmatrix}}_{=:\boldsymbol{\tau}_{\min,z}} \preceq \begin{bmatrix} \tau_z \\ \tau_z \end{bmatrix} \preceq \underbrace{\kappa\begin{bmatrix} 4u_{\max} - \hat{f} - 2\frac{|\hat{\tau}_y|}{l} \\ -4u_{\min} + \hat{f} - 2\frac{|\hat{\tau}_x|}{l} \end{bmatrix}}_{=:\boldsymbol{\tau}_{\max,z}}. \tag{46}$$

The applied yaw torque is therefore constrained to

$$\hat{\tau}_z \leftarrow \text{constrain}(\tau_z, \max(\boldsymbol{\tau}_{\text{min},z}), \min(\boldsymbol{\tau}_{\text{max},z})). \tag{47}$$

The resulting virtual control input $\hat{\boldsymbol{v}} = (\hat{f}, \hat{\boldsymbol{\tau}})$ is guaranteed to be attainable, i.e., $\hat{\boldsymbol{v}} \in \mathbb{V}$, and consequently a feasible control input $\boldsymbol{u} \in \mathbb{U}$ that produces $\hat{\boldsymbol{v}}$ can be found using (40).

B. Discussion

Achieving the desired yaw torque is given the least priority and is therefore the first one to be constrained. Since the reduced attitude dynamics are independent of the applied yaw torque, constraining the yaw torque does not compromise the stability of the reduced attitude dynamics and equivalently of the position dynamics. If the roll and pitch torques are constrained, then the stability of the reduced attitude dynamics is not guaranteed anymore. In case the attitude to be tracked is constant, i.e., $\boldsymbol{q}_\text{d}(t) = \boldsymbol{q}_\text{d}, \ \boldsymbol{\omega}_\text{d}(t) = 0$, then it is shown in [35] that the control allocation strategy preserves the stability of the reduced attitude for the simplified attitude control law with no feedback linearization term if $k_{\text{p},xy} < \tau_{\text{max},xy}/2$. However, the constraint on the proportional gain yields very slow controllers and experiments have shown that also much larger control gains work well in practice.

VI. Results

In this section, experimental results demonstrating the performance of the attitude controller and control allocation strategy are presented. A video of the experiments is available at `https://youtu.be/QH0oEXQuC6c`.

A. Experimental Setup

The experiments are carried out in Flying Machine Arena, an indoor aerial vehicle test bed at ETH Zurich [36]. A custom-built quadrocopter based on Ascending Technologies' Hummingbird [37] equipped with a PX4FMU[2] flight computer is used. The flight computer contains a 168 MHz Cortex M4F microcontroller that runs a state observer, a position controller, the proposed attitude controller, and the proposed control allocation algorithm, all at a rate of 1 kHz.

The state observer estimates the quadrocopter's position, velocity, and attitude and is driven by acceleration and angular velocity measurements obtained from an inertial measurement unit at a rate of 1 kHz. Every 20 ms, i.e., at 50 Hz, position and attitude measurements from an external motion capture system are sent to the quadrocopter through a low-latency wireless communication channel and fused with the quadrocopter's state estimate.

A position controller as presented in [36] is applied to track the desired position trajectories. The control loop for the vertical position is designed such that it responds to vertical position errors $p_{\text{e},z} = p_{\text{d},z} - p_z$ like a second-order system with time constant σ_z and damping ratio ζ_z

$$\sigma_z^2 \ddot{p}_{\text{e},z} + 2\zeta_z \sigma_z \dot{p}_{\text{e},z} + p_{\text{e},z} = 0. \tag{48}$$

[2]`https://pixhawk.org/modules/px4fmu`

TABLE I
Parameter Values

Symbol	Description	Value
u_min	Minimum rotor thrust	$0.2\,\text{N}$
u_max	Maximum rotor thrust	$3.4\,\text{N}$
κ	Rotor thrust-to-drag ratio	$1.6 \times 10^{-2}\,\text{Nm/N}$
l	Arm length of the quadrocopter	$0.17\,\text{m}$
m	Mass of the quadrocopter	$0.523\,\text{kg}$
J_{xx}	Moment of inertia about $\boldsymbol{e}_x^\mathcal{B}$	$2.3 \times 10^{-3}\,\text{kg}\,\text{m}^2$
J_{yy}	Moment of inertia about $\boldsymbol{e}_y^\mathcal{B}$	$2.3 \times 10^{-3}\,\text{kg}\,\text{m}^2$
J_{zz}	Moment of inertia about $\boldsymbol{e}_z^\mathcal{B}$	$4.6 \times 10^{-3}\,\text{kg}\,\text{m}^2$
$k_{\text{p},xy}$	Reduced attitude control P-gain	$3.286\,\text{N}\,\text{m}$
$k_{\text{d},xy}$	Reduced attitude control D-gain	$0.230\,\text{Nm/(rad/s)}$
$k_{\text{p},z}$	Yaw control P-gain	$0.197\,\text{N}\,\text{m}$
$k_{\text{d},z}$	Yaw control D-gain	$0.046\,\text{Nm/(rad/s)}$
σ_{xy}	x- and y-position time constant	$0.35\,\text{s}$
ζ_{xy}	x- and y-position damping ratio	0.95
σ_z	z-position time constant	$0.25\,\text{s}$
ζ_z	z-position damping ratio	0.8

Using the position dynamics (10) and the desired vertical position dynamics (48), the desired collective thrust f is computed to be

$$f = \frac{m(\ddot{p}_z + g)}{R_{33}(\boldsymbol{q})} \tag{49}$$

where the scalar $R_{33}(\boldsymbol{q})$ is the $(3,3)$ element of the rotation matrix $\boldsymbol{R}(\boldsymbol{q})$. Similarly, two control loops for the horizontal position of the quadrocopter are shaped to make the horizontal position errors behave like second-order systems with time constant σ_{xy} and damping ratio ζ_{xy}.

The attitude controller and control allocation strategy are implemented as presented in Sections IV and V, respectively. The reference attitude for the control law is obtained as follows: first, the commanded acceleration $\ddot{\boldsymbol{p}}$ is converted to a desired reduced attitude $\boldsymbol{q}_\text{d,red}$ using (10), with the rotation axis and angle given by

$$\varphi_\text{d,red} = \text{atan2}\left(\sqrt{\ddot{p}_x^2 + \ddot{p}_y^2}, \ddot{p}_z + g\right) \tag{50}$$

$$\boldsymbol{n}_\text{d,red} = \frac{1}{\sqrt{\ddot{p}_x^2 + \ddot{p}_y^2}} \begin{bmatrix} -\ddot{p}_y \\ \ddot{p}_x \\ 0 \end{bmatrix}. \tag{51}$$

The desired reduced attitude $\boldsymbol{q}_\text{d,red}$ is then rotated about its z-axis by a desired yaw angle $\varphi_\text{d,yaw}$ yielding the desired attitude $\boldsymbol{q}_\text{d}$

$$\boldsymbol{q}_\text{d} = \boldsymbol{q}_\text{d,yaw} \otimes \boldsymbol{q}_\text{d,red} \tag{52}$$

with $\boldsymbol{q}_\text{d,yaw} = (\cos(\varphi_\text{d,yaw}/2), 0, 0, \sin(\varphi_\text{d,yaw}/2))$. The quadrocopter's desired angular velocity and angular acceleration are computed accordingly based on the nominal maneuver and making use of the quadrocopter's differential flatness [6].

Table I summarizes the control parameters as well as the physical parameters of the quadrocopter used for the experiments. The rotor thrust limits and thrust-to-torque ratio were measured on a load cell and the quadrocopter's moments of inertia were obtained from a detailed CAD model.

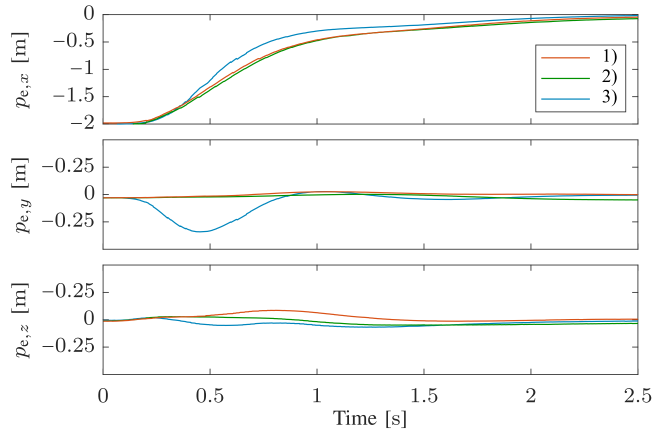

Fig. 6. Response in position for a set point change of 2 m in the x-direction and yaw change from zero to 135°. The experiment is carried out for 1) the proposed attitude controller, 2) the proposed attitude controller but without a jump in the yaw orientation, and for 3) the conventional quaternion-based PD controller (35).

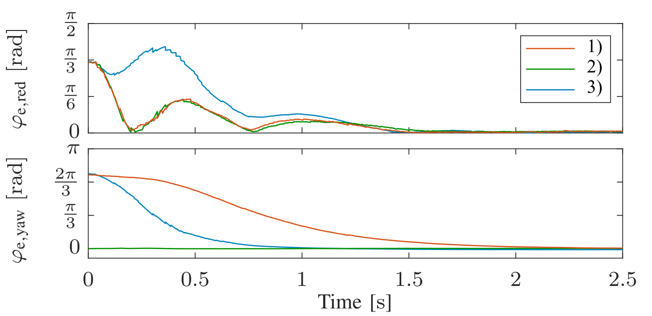

Fig. 7. Reduced attitude errors (top) and yaw errors (bottom) of the quadrocopter corresponding to the position responses depicted in Fig. 6. The response of the reduced attitude error is faster with the proposed attitude control law, i.e., for 1) and 2), than with a conventional controller 3), but at the expense of a slower response in the yaw error.

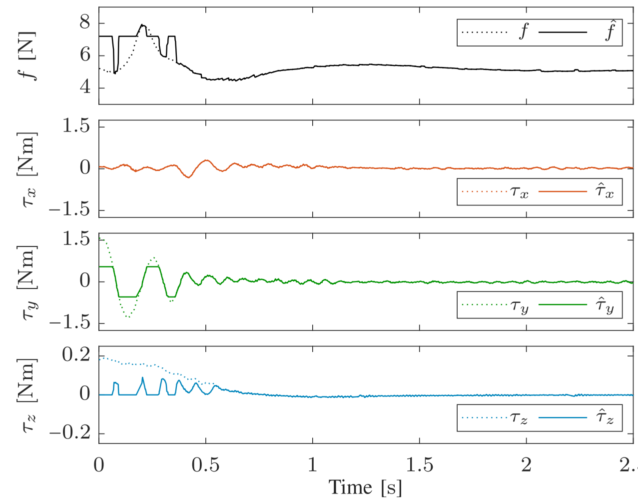

Fig. 8. Desired and applied virtual control inputs for experiment 1), denoted by dotted and solid lines, respectively. Initially, all rotor thrust resources are allocated to the desired roll and pitch torque. Once the desired roll and pitch torque are sufficiently small, the desired collective thrust and yaw torque are generated.

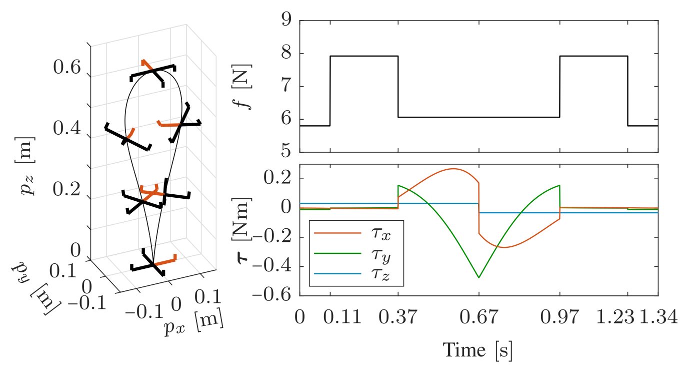

Fig. 9. Plot on the left depicts the flip maneuver that the quadrocopter is commanded to track (snapshots are taken at $t = \{0\ \mathrm{s}, 0.37\ \mathrm{s}, 0.5\ \mathrm{s}, 0.67\ \mathrm{s}, 0.84\ \mathrm{s}, 0.97\ \mathrm{s}\}$). The maneuver consists of six steps (see [38] for more details): slowly rotating outwards, accelerating upward, starting the rotation, stopping the rotation, decelerating, and slowly rotating back to the initial position. The plots on the right depict the nominal virtual control inputs during each of the six steps.

B. Experimental Results

1) Error Recovery: To test the attitude controller and control allocation strategy's response to large errors, the quadrocopter is commanded a 2 m set point change along the x-position and a jump in the desired yaw orientation from zero to 135°. The experiment is conducted three times: 1) with the proposed attitude controller; 2) with the proposed attitude controller but no set point change in yaw; and 3) with the conventional quaternion-based PD-controller (35). In the all three cases, the proposed control allocation strategy is applied.

Fig. 6 shows the position error responses and Fig. 7 shows the corresponding reduced attitude and yaw errors. Due to splitting the attitude error into a reduced attitude and yaw error, the proposed attitude controller moves the quadrocopter's thrust direction along the shortest angular path to the desired thrust direction, i.e., only in the inertial xz-plane, whereas the conventional control law rotates the thrust direction out of the inertial xz-plane and therefore causes a significant position error in the inertial y-direction. In addition, the proposed attitude controller yields identical position responses independent of the commanded yaw orientation, which can also be observed in the reduced attitude error response. This is due to the control allocation strategy that prioritizes the roll and pitch torques as well as the desired collective thrust over the yaw torque. The desired and applied virtual control inputs of experiment 1) are shown in Fig. 8. Although the large yaw error demands for a large yaw torque, the applied yaw torque is initially constrained to zero such that the demanded roll and pitch torque can be met, and the large yaw error has therefore no effect on the quadrocopter's position response.

2) Trajectory Tracking: The effectiveness of the control strategy for tracking aggressive maneuvers that require rotor thrusts close to their saturation limits is evaluated by flying a flip maneuver with a 180° yaw rotation as illustrated in Fig. 9. The flip maneuver is executed three times: 1) with the proposed attitude controller and control allocation strategy; 2) same as 1) but without a yaw rotation, and

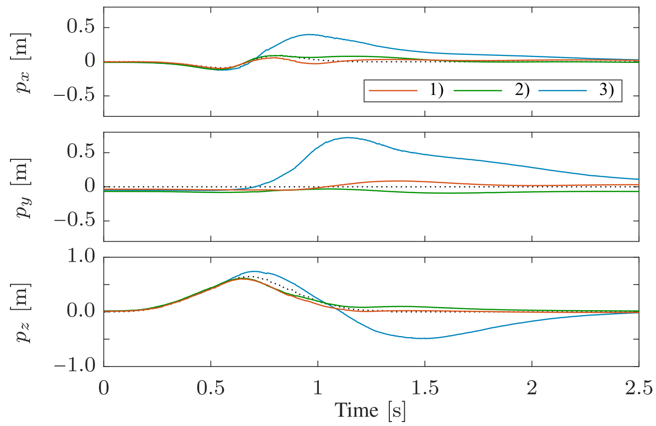

Fig. 10. Estimated position of the quadrocopter when tracking the flip maneuver of Fig. 9 with 1) the proposed attitude controller and control allocation strategy, 2) same as 1) but without a yaw rotation, and 3) with the conventional attitude controller (35) and thrust clipping. The dotted lines indicate the reference trajectories.

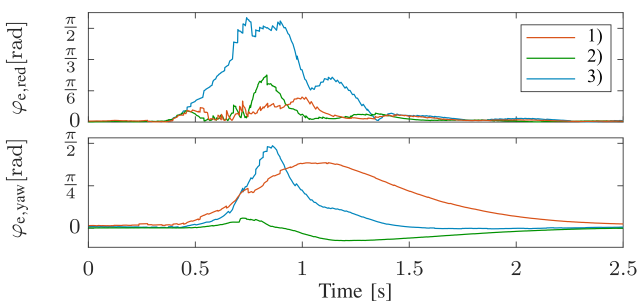

Fig. 11. Reduced attitude error and yaw error with respect to the nominal flip maneuver. Both control strategies track the flip maneuver well up to $t = 0.37$ s, at which point the required roll and pitch torque increase in order to initiate the flip. The proposed control strategy gives up the control of the yaw orientation and focuses on tracking the desired reduced attitude. Once the maneuver is finished and no more large roll and pitch torques are required, the yaw error is corrected.

3) with a conventional quaternion-based PD-controller (35) and with clipping the desired rotor thrusts at their saturation limits instead of the proposed control allocation strategy.

The quadrocopter's position, attitude and angular velocity during the flip maneuver are shown in Figs. 10–12, respectively. Up to 0.37 s, i.e., until the flip rotation begins, the performance of both control strategies is similar. After 0.37 s, large roll and pitch torques are required to track the nominal maneuver, yielding rotor thrusts that are close to their saturation limits, or even beyond due to disturbances and model uncertainties. The proposed control allocation strategy gives up yaw control in order to generate the desired roll and pitch torques and collective thrus more accurately and hence tracks the position trajectory better. Once the desired roll and pitch torques are small, the yaw error is corrected. Due to the decoupling of the reduced attitude from the yaw error dynamics, the yaw error that is caused by not generating the required yaw torque does not affect the tracking performance of the position trajectory (compare position

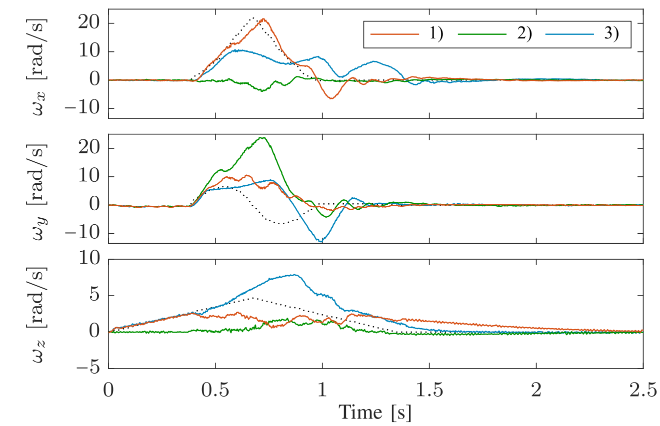

Fig. 12. Angular velocity of the quadrocopter during the flip maneuver for the three different cases 1), 2), and 3). The dotted lines denote the reference angular velocities for the flip maneuver with a 180° yaw rotation, i.e., for 1) and 3). It can be seen how the proposed control strategy prioritizes tracking the reduced attitude, i.e., the roll and pitch angular velocity, whereas the conventional strategy attempts to track all desired angular velocities.

response 1) and 2)). The conventional control strategy 3) tracks the desired yaw orientation more accurately for a longer period since none of the virtual control inputs is prioritized. However, because neither the desired roll and pitch torque nor the desired collective thrust are produced exactly, the reduced attitude error and the position error increase. Note that even with the proposed control allocation strategy, the conventional attitude controller is not able to track the position reference as well as 1) because the increasing yaw error would induce a position error in the y-direction (see Section VI-B1).

VII. Conclusion

This paper presented and analyzed an attitude control strategy for quadrocopters that is computationally lightweight and hence well suited for running on board quadrocopters at high rates. Based on separating the attitude error into a reduced attitude error and a yaw error, a model-based PD control law was derived that decoupled the reduced attitude error dynamics from the yaw error dynamics. The decoupling of the attitude error dynamics enabled the development of a simple control allocation strategy that prioritizes achieving the desired roll and pitch torque, which are required for the control of the quadrocopter's crucial thrust direction, over the collective thrust and yaw torque. Because achieving the desired yaw torque was given the least priority, it is the first to be constrained if the available control authority becomes scarce. Nonetheless, due to the decoupling of the attitude error dynamics, constraining the yaw torque was shown to not affect the reduced attitude error dynamics and equivalently the position dynamics.

The proposed control strategy was implemented on board a quadrocopter in order to evaluate its performance experimentally. The proposed control strategy showed to improve the quadrocopter's error recovery performance compared to controllers that are directly based on the full attitude error, as the quadrocopter's thrust direction is pushed along the shortest angular path toward the desired thrust direction and thereby induces less position error. The control allocation strategy in combination with the attitude control law was found to increase the position trajectory tracking performance for aggressive flight maneuvers where the desired rotor thrusts may exceed their saturation limits, since no control effort is wasted on tracking the non-crucial yaw angle and since the yaw error does not affect the reduced attitude error dynamics.

Although the proposed attitude control approach was experimentally verified to be favourable when recovering from large errors or tracking aggressive maneuvers, i.e., in scenarios where the rotor thrusts reach their saturation limits, stability of the control law has only been established in the absence of saturations. Future work thus includes the extension of the stability proof in order to take the rotor thrust saturations into account.

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